Mathematical misconceptions
-- we have an effective method for reducing their incidence but will the improvement persist?

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This paper discusses a continuation of the study previously conducted by these authors in which a strategy based on Piaget's notion of cognitive conflict was successfully employed to reduce the incidence of mathematical misconceptions in a group of tertiary students. A question which remained unanswered in the authors' previous study was "will the improvement persist or have we seen a short term gain which will disappear in time?" The students were tested again one year later to decide this issue.

Introduction

The variety and importance of mathematical misconceptions exhibited by students, and ways to reduce or eliminate them, has been studied by a number of researchers (Bell, 1982; Farrell, 1992; Margulies, 1993; Perso, 1992). Some of them believe that by considering the mathematical misconceptions which are exhibited by students, teaching techniques can be developed which aim at diagnosing and eliminating those misconceptions. Davis (1984) suggests that the number of different types of mathematical misconceptions is enormous, "and a complete list may not even be practical" (p. 335). Swedosh (1996) discussed the types, frequencies of occurrence, and possible reasons for the mathematical misconceptions commonly exhibited by mathematics students on entering tertiary mathematics subjects at the University of Melbourne (U. of M.) and at LaTrobe University (LaTrobe).

There have been various attempts to eliminate students' misconceptions in a wide range of fields. One approach which has met with some success is the "conflict teaching approach", based on Piaget's notion of cognitive conflict, in which a teacher and a learner discuss the inconsistencies in the learner's thinking so that the learner realises that the conceptions exhibited were inadequate or faulty and needed modification (Tirosh, 1990). Vinner (1990) argues that "there is no doubt that if inconsistencies in the students' thinking are drawn to their attention, it will help some of them to resolve some inconsistencies in a desirable way" (p. 97). The conflict teaching approach has been found to be effective in successfully resolving misconceptions relating to aspects of mathematics and physics (Stavy and Berkovitz, 1980; Strauss, 1972; Swan, 1983).

Swedosh and Clark (1997) reported on an experiment devised to test whether the conflict teaching strategy helped to reduce the frequency of mathematical misconceptions in a sample of mathematics students at the U. of M. The strategy proved to be extremely effective and the improvement greatly exceeded the expectations of the authors.

> "It is clear that by first challenging or undermining the misconception held by the students by showing the ridiculous outcomes which can flow from such 'rules', and then replacing the 'damaged' concept with the correct one, mathematical misconceptions can be, to a great extent, eliminated" (p. 498).

The study by Swedosh and Clark (1997) demonstrated that the conflict teaching approach was found to be effective in significantly reducing, and in some cases eliminating, mathematical misconceptions. The study did, however, raise some other questions, namely, would the strategy be as successful with less able students, would the strategy be as successful if the concepts being tested were embedded in more complex questions, and would the improvement persist or just be a short term phenomenon with students reverting to their previously held conceptions. It is this latter question which is examined in this study. The question of the effectiveness of this method with less able students will be addressed during Semester 1, 1998, when an experiment will be performed at the U. of M. using the conflict teaching approach on an appropriate group of students; the results of this study will be available at the MERGA Conference in July, 1998.

The importance of eliminating mathematical misconceptions, and arriving at a complete working understanding of basic concepts, is well documented and is especially important if students are intending to continue their studies in mathematics. The issues of a sound preparation for further studies and of increasing participation in post-secondary mathematics education have been officially recognised by the Victorian Government (Ministry of Education Victoria, 1984; Victorian Government, 1987) and in A National Statement on Mathematics for Australian Schools (Australian Education Council, 1990). One goal in the National Statement is that "as a result of learning mathematics in school, all students should possess sufficient command of mathematical expressions, representations and technology to continue to learn mathematics independently and collaboratively" (p. 18).

In many branches of mathematics, success at a particular level has been shown to depend heavily on students having previously achieved mastery of a limited number of basic concepts and being able to confidently use certain skills (Swedosh, 1996). In his paper, Swedosh also asserts that the preparedness of students to study tertiary mathematics is largely determined by the level of understanding of the basic mathematical concepts which are expected to be acquired in secondary school. Since subsequent learning of mathematics relies to such an extent on the mastery of pre-requisite knowledge, there has been a high level of interest amongst mathematics educators (teachers, lecturers, and tutors) regarding mathematical misconceptions, their frequencies of occurrence, and whether a method exists which effectively reduces these frequencies.

The issue of preparedness of students for tertiary mathematics takes on even greater significance when one considers the statements by Blyth and Calegari (1985) that "contrary to a widely held belief, 90% of all HSC students do apply for tertiary entrance. It is reasonable to infer from this that most students see HSC as a preparation for tertiary studies" (p. 312) and also, "statistics collected by the Mathematical Association of Victoria (MAV) show that 75% of all tertiary courses require a pass in HSC mathematics" (p. 312). These statements were made in 1985, and there is little to suggest that the situation has changed dramatically since then.

Given the negative impact that mathematical misconceptions can have on the mathematical future of students, and that the conflict teaching approach had been shown to be successful in reducing their frequencies, the next step was to investigate whether the improvement which had resulted would persist or whether the effect was temporary (that is, students would revert in due course to displaying the same misconceptions as they had previously). This study reports on our attempt to clarify this issue.

Methodology

The methodology for this study could be divided into two parts, the first part being the methodology from Swedosh and Clark (1997) in which the students were given a pre-test and a post-test, and the second part being the re-testing of a subset of the original group of students 12 months later. In each case, students who studied Applied Mathematics in their first year at the U. of M. in 1996 were involved. Each test was made up of questions similar to those used in earlier tests at the U. of M. and at LaTrobe (Worley, 1993) in which students had previously exhibited a high frequency of misconceptions (Swedosh, 1996) and some questions similar to those appearing in a list of questions in 'Algebraic Atrocities' (Margulies, 1993, p. 41). Each question was designed in such a way that if the student had a particular misconception, this would become apparent when considering the response of that student to the question.

The first test was called the pre-test as it was held prior to using a teaching strategy designed to reduce the frequency of misconceptions. This test consisted of 53 short answer questions. Students were given sufficient time to complete the test. Each response to each question on each student's paper was carefully examined to gain information on any misconceptions which had been exhibited. A tally was kept of the number of students who had attempted each question, how many had answered the question correctly, how many had exhibited a misconception, and how many had given another wrong answer. The questions in which students exhibited the highest frequency of misconceptions became the 'focus' questions of this study; there were 17 such questions. It was felt that these 17 questions could be completed by students in 10 minutes, and that this interval of time would not impinge too greatly on the lecturer's class time.

Two weeks after the pre-test, about twenty minutes of a lecture was dedicated to using the 'conflict teaching approach' in an attempt to eliminate or reduce the frequencies of these misconceptions and then to replace them with the correct concept. For example, 6.7% of students had exhibited the misconception that given $\frac{1}{x}-\frac{1}{b}=\frac{1}{a}$, then $x-b=a$ and therefore $x=a+b$. The approach used was to display to students an equation to which the result was self-evident, such as $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$. Using the misconception would lead to the absurd result that $2-3=6$. The correct procedure, in which a common denominator was used, was taught after showing the absurdity which stemmed from the misconception. The rationale in this method is that having seen that the previously held conception simply does not work, the student will happily discard it and embrace the correct concept.

Three weeks after the teaching session, the same class of students sat the post-test which comprised the 17 focus questions. The delay between the teaching and the post-test was important so that students were answering questions not based on remembering the actual examples they had been shown, but on what they now understood to be the correct concepts. Students were not informed in advance that there was to be a second test and so the results of this test could not be affected by any attempt to prepare for it.

For each question, a tally was again kept of the number of students attempting the question, how many had answered the question correctly, how many had exhibited a misconception, and how many had given another wrong answer, and a comparison was made between the frequency of the misconceptions on each question before and after the teaching session. A great improvement was found, with many less misconceptions being exhibited on the post-test. One question which begged for further investigation was whether this improvement would last, or whether, with time, students would revert to their previous, and often long held, misconceptions.

To address this issue, it was decided that about one year after the earlier testing, the students who had been part of the original study (Swedosh and Clark, 1997) would, if they could be contacted and were agreeable, be tested again using the same 17 questions. The students were contacted and asked to come in and sit a test to assist with a study being undertaken. Each student was sent a letter inviting them to participate in the study, and this was followed up by email, telephone, or further letters as appropriate. The students were not told what the test covered or what it hoped to achieve, only that we would appreciate them giving up ten minutes or so of their time to assist with our study. Some of these students were studying a variety of second year mathematics subjects and some were no longer studying any mathematics, but it was hoped that a representative number of those invited would accept due to the considerable good will established during the previous year. Of the 60 students who had been part of the original study, 28 accepted our invitation to be tested again one year later in October 1997. Readers of this paper are referred to the 1997 paper of Swedosh and Clark for a detailed discussion of the background of students who were part of the original study. Details provided include a description of the mathematics subjects studied at secondary school by those students, the assessment methods and grades awarded for those subjects, and an explanation of the Tertiary Entrance Ranking or TER. It can be seen from the statistics in Table 1 that the groups of 60 and 28 students have very similar average scores in mathematics and in their TERs and can be considered to be comprised of very comparable students.

Table 1

STUDENTS IN SAMPLE	AVERAGE SCORES TER	SPECIALIST MATHS CAT 1	CAT 2	CAT 3
60 students who sat pre-test and pre-test:	93.56	9.38	8.52	8.50
28 students who sat the test one year later:	93.34	9.39	8.71	8.43

The test

The 17 questions in the post-test and in the test sat one year later are shown below, and the most common misconception(s) are shown to the right of each question :

Simplify expressions 1-5 as fully as possible. Note that some expressions may not be able to be simplified. If this is the case, simply rewrite the expression in the space provided for the answer.

1.	$\frac{100!}{98!}$	$2!$
2.	$\left(\frac{1}{2}\right)^{-3}$	$\frac{1}{8}$
3.	$3^x \times 3^x$	9^{2x}; 3^{x^2}
4.	$2^x + 2^x$	4^x
5.	$a^b \times a^c$	a^{bc}

Solve equations 6-12 for x :

6.	$x^2 = 81$	$x = 9$
7.	$x^2 - 4x = 0$	$x = 4$
8.	$(x-1)(x^2 - 3x) = 0$	$x = 1$; $x = 1,3$
9.	$x^2 = x$	$x = 1$
10.	$\frac{x^2 - 1}{x - 1} = 0$	$x = \pm 1$; $x = 1$
11.	$\frac{1}{x} - \frac{1}{b} = \frac{1}{a}$	$x = a + b$
12.	$\log_e x = \log_e t - t + c$, c constant	$x = t - e^t + e^c$

13. Solve for x : $2x+4<5x+10$ $x<-2;\ x>2$

14. Factorise $(2x+y)^2-x^2$ $3x^2+4xy+y^2$

15. Given that $\sin\frac{\pi}{6}=\frac{1}{2}$, evaluate $\sin\frac{7\pi}{6}$ $\frac{1}{2}$

16. Given that $\sin A=\frac{3}{5}$ and that $\frac{\pi}{2}<A<\pi$, evaluate $\cos A$. $\frac{4}{5}$

For question 17, indicate whether the statement is True or False. (circle one)

17. $k\left(50-\frac{x}{5}\right)(80-2x)=k(250-x)(40-x)$

T / F True

The teaching strategy

Teaching designed to reduce or eliminate the misconceptions exhibited in the pre-test was employed two weeks after the pre-test (Swedosh and Clark, 1997). The concepts arising in questions which were to be on the post-test and on the test sat one year later were specifically targeted. The strategy involved demonstrating that the misconception would lead to a ridiculous conclusion and used mainly numerical examples. Having shown the students that a conception was faulty and in need of replacement, the correct concept was illustrated using slightly different examples to the questions on the post-test (different numbers, etc.) so that students could not simply remember what they had been shown, but had to use the concept correctly to answer each question. Some examples of how the strategy was used and how the correct concept was illustrated are shown below.

$\frac{4!}{2!}=\frac{24}{2}=12,\ (4-2)!=2!=2;\qquad \frac{4!}{2!}\neq(4-2)!$

$3^2\times3^2=9\times9=81;\ \ 9^4=9\times9\times9\times9=81^2;\ \ 3^2\times3^2\neq9^4$

$2^3+2^3=8+8=16;\ \ 4^3=4\times4\times4=64;\ \ 2^3+2^3\neq4^3$

$x^2=9,\ x^2-9=0,\ (x-3)(x+3)=0,\ x=\pm3$

$\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}=x+2$ iff $x\neq2$; $\frac{0}{0}$ does not exist.

$\frac{1}{2}-\frac{1}{3}=\frac{1}{6};\ \ 2-3\neq6$

$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2};\ \sin\frac{4\pi}{3}=\sin\left(\pi+\frac{\pi}{3}\right)=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$ as $\frac{4\pi}{3}$ is in the third quadrant.

Results

The two tables below summarise the answers to the 17 test questions. Tables 2 and 3 show the frequencies and percentages respectively of each category of answer given in the pre-test, the post-test, and after one year. 'Q' is the question number.

Table 2

Q	Correct			Misconception			Other wrong			No attempt		
	Pre	Post	1 year	Pre	Post	1 year	Pre	Post	1 year	Pre	Post	1 year
	n=60		n=28	n=60		n=28	n=60		n=28	n=60		n=28
1	52	58	27	5	0	0	2	2	1	1	0	0
2	56	60	27	4	0	1	0	0	0	0	0	0
3	54	56	27	6	2	0	0	1	1	0	1	0
4	39	47	22	8	2	2	12	9	4	1	2	0
5	52	57	24	7	2	3	1	0	1	0	1	0
6	49	60	26	11	0	2	0	0	0	0	0	0
7	53	59	26	5	0	1	2	1	1	0	0	0
8	53	57	27	6	1	1	1	2	0	0	0	0
9	28	57	24	30	2	4	2	1	0	0	0	0
10	46	44	24	12	9	1	2	6	3	0	1	0
11	56	57	27	4	2	0	0	0	1	0	1	0
12	36	54	22	15	2	5	4	2	1	5	2	0
13	45	50	23	7	4	3	4	6	1	4	0	1
14	41	55	21	7	0	1	10	2	5	2	3	1
15	51	51	26	7	6	0	2	2	2	0	1	0
16	27	50	20	23	5	5	4	4	3	6	1	0
17	44	54	24	10	4	4	0	0	0	6	2	0

Table 3

Of those who attempted the question

Q	% Correct			% Misconception			% Other error		
	Pre	Post	1 year	Pre	Post	1 year	Pre	Post	1 year
1	88.1	96.7	96.4	8.5	0.0	0.0	3.4	3.3	3.6
2	93.3	100.0	96.4	6.7	0.0	3.6	0.0	0.0	0.0
3	90.0	94.9	96.4	10.0	3.4	0.0	0.0	1.7	3.6
4	66.1	81.0	78.6	13.6	3.4	7.1	20.3	15.5	14.3
5	86.7	96.6	85.7	11.7	3.4	10.7	1.7	0.0	3.6
6	81.7	100.0	92.9	18.3	0.0	7.1	0.0	0.0	0.0
7	88.3	98.3	92.9	8.3	0.0	3.6	3.3	1.7	3.6
8	88.3	95.0	96.4	10.0	1.7	3.6	1.7	3.3	0.0
9	46.7	95.0	85.7	50.0	3.3	14.3	3.3	1.7	0.0
10	76.7	74.6	85.7	20.0	15.3	3.6	3.3	10.2	10.7
11	93.3	96.6	96.4	6.7	3.4	0.0	0.0	0.0	3.6
12	65.5	93.1	78.6	27.3	3.4	17.9	7.3	3.4	3.6
13	80.4	83.3	85.2	12.5	6.7	11.1	7.1	10.0	3.7
14	70.7	96.5	77.8	12.1	0.0	3.7	17.2	3.5	18.5
15	85.0	86.4	92.9	11.7	10.2	0.0	3.3	3.4	7.1
16	50.0	84.7	71.4	42.6	8.5	17.9	7.4	6.8	10.7
17	81.5	93.1	85.7	18.5	6.9	14.3	0.0	0.0	0.0
Total	78.6	92.1	88.0	16.8	4.1	7.0	4.6	3.8	5.1

The results indicate that overall the proportion of misconceptions dropped substantially from the pre-test (16.8%) to the post-test (4.1%), and that one year later there was some reversion (7.0%) as expected. Nevertheless, a marked improvement was seen from the first test to the final test both in terms of the decrease in the proportion of misconceptions and the proportion of correct answers given (78.6% to 88.0%).

When considering the proportion of misconceptions on a question by question basis, on every question there was a substantial improvement from the pre-test to the post-test except for questions 10 and 15 in which the improvement was slight; possible reasons are given in Swedosh and Clark (1997). One year later, most of the questions show the same pattern; an improvement from the pre-test to the post-test, followed by a slight 'drift' one year later, though not back to the original levels. The exceptions were questions 1, 3, 10, 11 and 15. Question 10 showed no drift, and the results for the other four questions actually showed further improvement between the post-test and the test held one year later. The reasons for these unexpected improvements have been considered by the authors and are most likely to be due to either chance (28 of a total of 60 students sat the final test), a further year of exposure to mathematics for many of the sample in which they saw the correct concept during lectures and tutorials, or perhaps for some topics or questions it takes longer than a few weeks for the understanding to 'sink in'.

Limitations of the Study

As stated in Swedosh and Clark (1997), it is important to keep the findings of this study in context, as it is not clear that it is valid to generalise them beyond the scope of the cohort examined. The strategy used, based on cognitive conflict, has been found to be very useful with very bright students who were strong mathematically, but any assertion made about other categories of students might not be appropriate. The improvement for students whose backgrounds were not as good may not be as dramatic or may not occur. The reason for this conjecture is two-fold: the students involved in this study were capable of recognising the inconsistencies in their previous thinking and then learning the correct concepts quickly; and these students demonstrated during discussions with the authors that they were embarrassed by making the errors that they did on what they considered to be material of such an elementary nature, and therefore had a strong desire to remedy the situation. Both of these qualities are likely to be more pronounced with better students. As previously stated, the effectiveness of this method with less able students will be investigated by using this strategy on an appropriate group of students during Semester 1, 1998 at the U. of M.

There is also a possibility if the concept being considered was embedded in some longer and less transparent question, misconceptions may be more likely to recur.

Conclusions

The teaching strategy used, based on the notion of cognitive conflict, was extremely effective and the improvement from the pre-test to the post-test and to the test held one year later certainly exceeded the expectations of the authors. It was expected that there would be some degree of drift when students were tested one year later, but the bulk of the improvement persisted. Swedosh and Clark (1997, p. 499) stated that

> "It is clear that by first challenging or undermining the misconception held by the students by showing the ridiculous outcomes which can flow from such 'rules', and then replacing the 'damaged' concept with the correct one, mathematical misconceptions can be, to a great extent, eliminated."

It is also clear that the improvement seen due to this strategy is much more than a short term quick fix with substantial benefits still evident one year later.

This strategy may not be as effective with students who are less mathematically able or perhaps less able overall; this will be the subject of further research in Semester 1, 1998, and this research will be available at the MERGA Conference in July, 1998. Also, if the questions used were more complex and the concepts were therefore tested more subtly, the level of improvement may be quite different.

This study set out to ascertain whether the strategy which was used to eliminate mathematical misconceptions and which was judged to be a remarkable success in the short term (Swedosh and Clark, 1997), would continue to have a positive effect in the longer term in that some or most of the benefit would persist. The results show that while a small proportion of the improvement diminished, a large improvement was still evident one year later and most of the benefit to students had been retained.

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